# Hermite Splines in Lie Groups as Products of Geodesics

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## 1 Introduction

### 1.1 Goal

This document defines a curve in the Lie group $G$ parametrized by time and by structural parameters in the associated Lie algebra $\mathfrak{g}$, and shows how to compute its differentials by all parameters.

The curve is expressed primarily in the form

$$S_n : \mathbb{R} \times \mathfrak{g}^n \to G \quad (1)$$

$$S_n(t) = \exp(a_n \cdot p_n) \cdot \exp(a_{n-1} \cdot p_{n-1}) \cdot \ldots \cdot \exp(a_1 \cdot p_1) \quad (2)$$

Having established notation and some useful identities, we show how to compute differentials of the product of exponentials, and then describe how to achieve boundary conditions on the curve's values and derivatives by choosing $a_i$ and $p_i$ appropriately.

The result is a form suitable for expressing an individual segment of a Hermite spline in the group, complete with all the structural differentials useful for fitting the spline to data. Chaining such segments with shared boundary conditions on neighbors yields a $C^2$ trajectory in the group, permitting parametric regression against observations involving the value, first, and second time derivatives of the trajectory.

The formulation is valid for any Lie group, but as a motivating example, consider using such a spline to model a rigid body trajectory in $SE(3)$, and adjusting the spline parameters to maximize the likelihood of noisy observations from a gyroscope and accelerometer on the body.

### 1.2 Notation

Elements of the Lie group $G$ are denoted with upper-case letters, while elements of the associated Lie algebra $\mathfrak{g}$ are denoted with lower-case letters.

The identity matrix is written $I$.

The adjoint representation in the group is written $\mathrm{Ad}[]$, while the adjoint representation in the algebra is written $\mathrm{ad}[]$.

The exponential map is written exp. It maps elements of $\mathfrak{g}$ to $G$, and elements of $\mathrm{ad}[\mathfrak{g}]$ to $\mathrm{Ad}[G]$. Its inverse is the logarithm log.

*Corrected left-hand-sides of equations 43-45 from $S'$ to $S''$.

## 2 Identities

This section enumerates identities useful in the rest of the document, and can be skipped if the reader is familiar with representations of Lie groups and their properties.

Throughout this section, let $x, y \in \mathfrak{g}$ and $X, Y \in G$.

### 2.1 Basic properties of $\mathrm{Ad}\left[\right]$,$\mathrm{ad}\left[\right]$, and $\exp\left(\right)$

The adjoint operator in $\mathfrak{g}$ is anti-commutative:

$$\mathrm{ad}\left[x\right] \cdot y = -\mathrm{ad}\left[y\right] \cdot x \tag{3}$$

The exponential map and the adjoint representation commute:

$$\exp\left(\mathrm{ad}\left[x\right]\right) = \mathrm{Ad}\left[\exp\left(x\right)\right] \tag{4}$$

The adjoint transforms right tangent vectors to left tangent vectors:

$$X \cdot \exp\left(y\right) = \exp\left(\mathrm{Ad}\left[X\right] \cdot y\right) \cdot X \tag{5}$$

The adjoint is a group homomorphism:

$$\mathrm{Ad}\left[X \cdot Y\right] = \mathrm{Ad}\left[X\right] \cdot \mathrm{Ad}\left[Y\right] \tag{6}$$

### 2.2 $\mathrm{Ad}\left[X\right] \cdot \mathrm{ad}\left[y\right] = \mathrm{ad}\left[\mathrm{Ad}\left[X\right] \cdot y\right] \cdot \mathrm{Ad}\left[X\right]$

Using the above, we can show that the adjoint in the group can transform the adjoint in the algebra from right to left as well. Consider $t \in \mathbb{R}$.

$$X \cdot \exp\left(t \cdot y\right) = \exp\left(\mathrm{Ad}\left[X\right] \cdot t \cdot y\right) \cdot X \tag{7}$$

First send all factors to the adjoint representation:

$$\mathrm{Ad}\left[X\right] \cdot \mathrm{Ad}\left[\exp\left(t \cdot y\right)\right] = \mathrm{Ad}\left[\exp\left(\mathrm{Ad}\left[X\right] \cdot t \cdot y\right)\right] \cdot \mathrm{Ad}\left[X\right] \tag{8}$$

Next re-order the application of the exponential map and the adjoint operator:

$$\mathrm{Ad}\left[X\right] \cdot \exp\left(\mathrm{ad}\left[t \cdot y\right]\right) = \exp\left(\mathrm{ad}\left[\mathrm{Ad}\left[X\right] \cdot t \cdot y\right]\right) \cdot \mathrm{Ad}\left[X\right] \tag{9}$$

Then use the linearity of the adjoint operator in the algebra:

$$\mathrm{Ad}\left[X\right] \cdot \exp\left(t \cdot \mathrm{ad}\left[y\right]\right) = \exp\left(t \cdot \mathrm{ad}\left[\mathrm{Ad}\left[X\right] \cdot y\right]\right) \cdot \mathrm{Ad}\left[X\right] \tag{10}$$

Finally, differentiate by $t$ at $t = 0$:

$$\mathrm{Ad}\left[X\right] \cdot \mathrm{ad}\left[y\right] = \mathrm{ad}\left[\mathrm{Ad}\left[X\right] \cdot y\right] \cdot \mathrm{Ad}\left[X\right] \tag{11}$$

## 2.3 Differentials in $G$ and the adjoint

We define differentials in the group in terms of left updates, so that the differential is a linear mapping from the parameter(s) of interest $\alpha$ to the algebra $\mathfrak{g}$:

$$\frac{\partial X(\alpha)}{\partial \alpha} \equiv \frac{\partial \left( \log \left( X(\alpha+\delta) \cdot X(\alpha)^{-1} \right) \right)}{\partial \delta} \Big|_{\delta=0} \tag{12}$$

Differentials by left updates are then simple. Consider a linear operator on the algebra,$L$:

$$\frac{\partial (\exp(L \cdot \epsilon) \cdot X)}{\partial \epsilon} \Big|_{\epsilon=0} = L \tag{13}$$

This makes differentials of products straightforward:

$$\frac{\partial (X \cdot Y)}{\partial Y} = \mathrm{Ad}[X] \tag{14}$$

$$\frac{\partial (X \cdot Y)}{\partial X} = I \tag{15}$$

We can also differentiate transformations of algebra vectors by the adjoint:

$$\frac{\partial (\mathrm{Ad}[\exp(\delta) \cdot X] \cdot y)}{\partial \delta} \Big|_{\delta=0} = \frac{\partial (\mathrm{Ad}[\exp(\delta)] \cdot \mathrm{Ad}[X] \cdot y)}{\partial \delta} \tag{16}$$

$$= \frac{\partial (\exp(\mathrm{ad}[\delta]) \cdot \mathrm{Ad}[X] \cdot y)}{\partial \delta} \tag{17}$$

$$= \frac{\partial (\mathrm{ad}[\delta] \cdot \mathrm{Ad}[X] \cdot y)}{\partial \delta} \tag{18}$$

$$= -\frac{\partial \mathrm{ad}[\mathrm{Ad}[X] \cdot y] \cdot \delta}{\partial \delta} \tag{19}$$

$$= -\mathrm{ad}[\mathrm{Ad}[X] \cdot y] \tag{20}$$

# 3 $S(t)$ and its Time Derivatives

## 3.1 Function of time

We define the product recursively:

$$a_i : \mathbb{R} \to \mathbb{R} \tag{21}$$

$$p_i \in \mathfrak{g} \tag{22}$$

$$S_i : \mathbb{R} \times \mathfrak{g} \to G \tag{23}$$

$$S_i(t) \equiv \exp(a_i(t) \cdot p_i) \cdot S_{i-1}(t) \tag{24}$$

$$S_1(t) \equiv \exp(a_1(t) \cdot p_1) \tag{25}$$

$S_n(t)$ is then evaluated by left-multiplying each exponential term from $i = 1$ to $n$.

### 3.2 First derivative by time

Consider the exponential factor $A_i(t) \equiv \exp(a_i(t) \cdot p_i)$:

$$
\begin{aligned}
A_i(t+\epsilon) &= \exp\left(\left(a_i(t) + \epsilon \cdot a_i'(t) + O\left(\epsilon^2\right)\right) \cdot p_i\right) &(26)\\
&= \exp\left(\left(\epsilon \cdot a_i'(t) + O\left(\epsilon^2\right)\right) \cdot p_i\right) \cdot \exp(a_i(t) \cdot p_i) &(27)\\
&= \exp\left(\left(\epsilon \cdot a_i'(t) + O\left(\epsilon^2\right)\right) \cdot p_i\right) \cdot A_i(t) &(28)\\
A_i'(t) &= \frac{\partial A_i(t+\epsilon)}{\partial \epsilon}|_{\epsilon=0} &(29)\\
&= a_i'(t) \cdot p_i &(30)
\end{aligned}
$$

The first time derivative $S_n'(t)$ of $S_n(t)$ is then straightforward to express recursively using the product and chain rules:

$$
\begin{aligned}
S_i(t) &= A_i(t) \cdot S_{i-1}(t) &(31)\\
S_i'(t) &= A_i'(t) \cdot S_{i-1}(t) + \mathrm{Ad}[A_i(t)] \cdot S_{i-1}'(t) &(32)\\
&= a_i'(t) \cdot p_i + \mathrm{Ad}[A_i(t)] \cdot S_{i-1}'(t) &(33)
\end{aligned}
$$

### 3.3 Second derivative by time

The second derivative is the partial derivative by $t$ of Eq.33:

$$
\begin{aligned}
S_i''(t) &= \frac{\partial S_i'(t)}{\partial t} &(34)\\
&= a_i''(t) \cdot p_i + \mathrm{Ad}[A_i(t)] \cdot S_{i-1}''(t) - \mathrm{ad}\left[\mathrm{Ad}[A_i(t)] \cdot S'_{i-1}(t)\right] \cdot A_i'(t) &(35)\\
&= a_i''(t) \cdot p_i + \mathrm{Ad}[A_i(t)] \cdot S_{i-1}''(t) + \mathrm{ad}\left[A_i'(t)\right] \cdot \mathrm{Ad}[A_i(t)] \cdot S'_{i-1}(t) &(36)
\end{aligned}
$$

## 4 Structural Differentials for Regression

In order to perform regression of the model against observations, we'll need differentials of $S$, $S'$, and $S''$ by the parameters $\{p_i\}$.

To this end, we'll require the differential of the exponential map, described in detail in another document ('Differential of the Exponential Map'):

$$
D_{\exp}[x] \equiv \frac{\partial \exp(x)}{\partial x} \equiv \frac{\partial \log(\exp(x+\delta) \cdot \exp(-x))}{\partial \delta} \quad (37)
$$

For many Lie groups of interest, $D_{\exp}[x]$ can be computed in closed form.

For convenience, we label the differential of the exponential factor $A_i$ of $S_i$ by its argument:

$$
D_i \equiv D_{\exp}[a_i(t) \cdot p_i] \quad (38)
$$

The time parameter $t$ is omitted in most of the equations below to avoid clutter.

### 4.1 Structural differentials of $S(t)$

Eq.24 is differentiated using the chain rule and Eq.14:

$$
\begin{aligned}
\frac{\partial S_i}{\partial p_i} &= \frac{\partial A_i}{\partial p_i} && (39)\\
&= D_{\exp}\left[a_i \cdot p_i\right] \cdot a_i && (40)\\
&= D_i \cdot a_i && (41)\\
\frac{\partial S_i}{\partial p_j} &= \begin{cases} \operatorname{Ad}\left[A_i\right] \cdot \left(\frac{\partial S_{i-1}}{\partial p_j}\right) & j < i \\ 0 & j > i \end{cases} && (42)
\end{aligned}
$$

### 4.2 Structural differentials of $S'(t)$

Eq.33 is differentiated using the product rule and Eq.20:

$$
\begin{aligned}
\frac{\partial S'_i}{\partial p_i} &= \left(\frac{\partial A'_i}{\partial p_i}\right) - \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S'_{i-1}\right] \cdot \left(\frac{\partial A_i}{\partial p_i}\right)\\
&= {a'}_i \cdot I - \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S'_{i-1}\right] \cdot \left(\frac{\partial A_i}{\partial p_i}\right)\\
\frac{\partial S'_i}{\partial p_j} &= \begin{cases} \operatorname{Ad}\left[A_i\right] \cdot \left(\frac{\partial S'_{i-1}}{\partial p_j}\right) & j < i \\ 0 & j > i \end{cases}
\end{aligned}
$$

### 4.3 Structural differentials of $S''(t)$

Eq.36 is differentiated using the product rule and Eq.20:

$$
\begin{aligned}
\frac{\partial S''_i}{\partial p_i} &= a''_i \cdot I - \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S''_{i-1}\right] \cdot \left(\frac{\partial A_i}{\partial p_i}\right) - \operatorname{ad}\left[A'_i\right] \cdot \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S'_{i-1}\right] \cdot \left(\frac{\partial A_i}{\partial p_i}\right) - a'_i \cdot \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S'_{i-1}\right] && (43)\\
&= a''_i \cdot I - \left(\operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S''_{i-1}\right] + \operatorname{ad}\left[A'_i\right] \cdot \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S'_{i-1}\right]\right) \cdot \left(\frac{\partial A_i}{\partial p_i}\right) - a'_i \cdot \operatorname{ad}\left[\operatorname{Ad}\left[A_i\right] \cdot S'_{i-1}\right] && (44)\\
\frac{\partial S''_i}{\partial p_j} &= \begin{cases} \operatorname{Ad}\left[A_i\right] \cdot \left(\frac{\partial S''_{i-1}}{\partial p_j}\right) + \operatorname{ad}\left[A'_i\right] \cdot \operatorname{Ad}\left[A_i\right] \cdot \left(\frac{\partial S'_{i-1}}{\partial p_j}\right) & j < i \\ 0 & j > i \end{cases} && (45)
\end{aligned}
$$

### 4.4 Efficient computation of structural differentials

Directly applying the recursive definitions above requires $O\left(n^2\right)$ time to compute structural differentials $\partial S_n$, $\partial S'_n$, and $\partial S''_n$.

By reordering the recursion from left to right and accumulating products and sums, these differentials can all be computed in $O(n)$ time:

$$L_n = I \tag{46}$$
$$l_n = 0 \tag{47}$$
$$\frac{\partial S_n}{\partial p_i} = \mathrm{Ad}\left[L_i\right] \cdot \left(\frac{\partial S_i}{\partial p_i}\right) \tag{48}$$
$$\frac{\partial S'_n}{\partial p_i} = \mathrm{Ad}\left[L_i\right] \cdot \left(\frac{\partial S'_i}{\partial p_i}\right) \tag{49}$$
$$\frac{\partial S''_n}{\partial p_i} = \mathrm{Ad}\left[L_i\right] \cdot \left(\frac{\partial S''_i}{\partial p_i}\right) + \mathrm{ad}\left[l_i\right] \cdot \left(\frac{\partial S'_n}{\partial p_i}\right) \tag{50}$$
$$L_{i-1} = L_i \cdot A_i \tag{51}$$
$$l_{i-1} = l_i + \mathrm{Ad}\left[L_i\right] \cdot A'_i \tag{52}$$

Note that Eq. 50 makes use of the identity given in Eq.11.

# 5 Constructing a Spline Segment with Boundary Conditions

By choosing $a_i$ and $p_i$ appropriately, we can construct a $C^\infty$ curve segment $B(t)$ with specified boundary values and derivatives (sometimes called a *Hermite spline*). Such segments can then be concatenated to form a $C^2$ spline in the group by matching the boundary conditions (value, first, and second time derivatives) between neighboring segments.

The time interval in each segment is, without loss of generality, assumed to be the unit interval $[0,1]$. Any other interval can be remapped to this by an affine transformation (as can the differentials).

The desired boundary conditions of the segment are specified as group elements for the function values at the boundaries, and algebra elements for the time derivatives:

$$B(0) = B_0 \in G \tag{53}$$
$$B(1) = B_1 \in G \tag{54}$$
$$B'(0) = d_0 \in \mathfrak{g} \tag{55}$$
$$B'(1) = d_1 \in \mathfrak{g} \tag{56}$$
$$B''(0) = h_0 \in \mathfrak{g} \tag{57}$$
$$B''(1) = h_1 \in \mathfrak{g} \tag{58}$$

## 5.1 Segment definition

In order to have enough degrees of freedom to achieve the boundary conditions, the Hermite spline segment is defined as a product of six terms, one of which is constant over time:

$$B(t) = S_5(t) \cdot B_0 \tag{59}$$

We choose the $\{a_i\}$ to be quintic polynomials with values and derivatives at 0 and 1 that make satisfying the boundary conditions of $B(t)$ trivial.

$$a_5(t) = \frac{1}{2}t^5 - t^4 + \frac{1}{2}t^3 \tag{60}$$

$$a_4(t) = -\frac{1}{2}t^5 + \frac{3}{2}t^4 - \frac{3}{2}t^3 + \frac{1}{2}t^2 \tag{61}$$

$$a_3(t) = -3t^5 + 7t^4 - 4t^3 \tag{62}$$

$$a_2(t) = -3t^5 + 8t^4 - 6t^3 + t \tag{63}$$

$$a_1(t) = 6t^5 - 15t^4 + 10t^3 \tag{64}$$

or equivalently:

$$\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ a_5(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 10 & -15 & 6 \\ 1 & 0 & -6 & 8 & -3 \\ 0 & 0 & -4 & 7 & -3 \\ 0 & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \end{pmatrix} \tag{65}$$

These polynomials are constructed to satisfy the following boundary conditions:

$$a_5(0) = a_4(0) = a_3(0) = a_2(0) = a_1(0) = 0 \tag{66}$$

$$a_5(1) = a_4(1) = a_3(1) = a_2(1) = 0 \tag{67}$$

$$a_1(1) = 1 \tag{68}$$

$$a_2'(0) = 1 \tag{69}$$

$$a_5'(0) = a_4'(0) = a_3'(0) = a_1'(0) = 0 \tag{70}$$

$$a_5'(1) = a_4'(1) = a_2'(1) = a_1'(1) = 0 \tag{71}$$

$$a_3'(1) = 1 \tag{72}$$

$$a_5''(0) = a_3''(0) = a_2''(0) = a_1''(0) = 0 \tag{73}$$

$$a_4''(0) = 1 \tag{74}$$

$$a_4''(1) = a_3''(1) = a_2''(1) = a_1''(1) = 0 \tag{75}$$

$$a_5''(1) = 1 \tag{76}$$

Thus each polynomial selects for the value, first time derivative, or second time derivative at one of the boundaries. The parameters of $S_5(t)$ are then straightforward to assign:

$$p_5 = h_1 \tag{77}$$

$$p_4 = h_0 \tag{78}$$

$$p_3 = d_1 \tag{79}$$

$$p_2 = d_0 \tag{80}$$

$$p_1 = \log\left(B_1 \cdot B_0^{-1}\right) \tag{81}$$

By Eq.66 all of the factors of $S_5(0)$ are the identity, so we satisfy the first boundary condition trivially:

$$B(0) = B_0 \tag{82}$$

By Eq.67 and 68 the second boundary condition is satisfied:

$$\begin{aligned} B(1) &= S_5(1)\cdot B_0 && (83)\\ &= \left(B_1\cdot B_0^{-1}\right)\cdot B_0 && (84)\\ &= B_1 && (85)\end{aligned}$$

Similar inspection of $S_5'(t)$ and $S_5''(t)$ shows that all four of the boundary conditions on the derivatives $\{d_0, d_1, h_0, h_1\}$ are satisfied.

## 5.2 Time derivatives of $B(t)$

Because the $B_0$ factor is constant over time, the time derivatives of $B(t)$ are taken directly from those of $S_5(t)$, described in detail above:

$$\begin{aligned} B'(t) &= S_5'(t) && (86)\\ B''(t) &= S_5''(t) && (87)\end{aligned}$$

## 5.3 Structural differentials of $B(t)$

The differentials of $B(t)$ by the boundary parameters $\{B_0, B_1, d_0, d_1, h_0, h_1\}$ follow from the differentials of $S_5(t)$ by $p_i$, along with the chain rule.

First we make explicit the parametric perturbations of $B_0$and $B_1$:

$$\begin{aligned} \tilde{B}_0 &\equiv \exp(\epsilon_0)\cdot B_0 && (88)\\ \tilde{B}_1 &\equiv \exp(\epsilon_1)\cdot B_1 && (89)\\ \tilde{p}_1 &\equiv \log\left(\tilde{B}_1\cdot \tilde{B}_0^{-1}\right) && (90)\\ &= \log\left(\exp(\epsilon_1)\cdot B_1\cdot B_0^{-1}\cdot\exp(-\epsilon_0)\right) && (91)\\ \tilde{B}(t) &= \exp(a_5(t)\cdot p_5)\cdot\exp(a_4(t)\cdot p_4)\cdot\exp(a_3(t)\cdot p_3)\cdot\exp(a_2(t)\cdot p_2)\cdot\exp(a_1(t)\cdot \tilde{p}_1)\cdot\tilde{B}_0 && (92)\end{aligned}$$

Note that changes to the initial value parameter $B_0$ (via $\epsilon_0$) affect $B(t)$ both directly and through the resulting modification of $p_1$, while changes to $B_1$ (via $\epsilon_1$) affect $B(t)$ only through the modification of $p_1$.

To differentiate $p_1$ by its arguments, we employ the property that the differential of the inverse is the inverse of the differential (see document 'Differential of the Exponential Map') :

$$\begin{aligned} \frac{\partial p_1}{\partial B_1} \equiv \frac{\partial \tilde{p}_1}{\partial \epsilon_1}|_{\epsilon_i=0} &= \left(D_{\exp}[p_1]\right)^{-1} && (93)\\ \frac{\partial p_1}{\partial B_0} \equiv \frac{\partial \tilde{p}_1}{\partial \epsilon_0}|_{\epsilon_i=0} &= \left(D_{\exp}[p_1]\right)^{-1}\cdot -\mathrm{Ad}\left[B_1\cdot B_0^{-1}\right] && (94)\end{aligned}$$

Finally, the structural differentials of $B(t)$ follow by the chain rule:

$$\frac{\partial B(t)}{\partial C_0} = \left(\frac{\partial S_5(t)}{\partial p_1}\right) \cdot \left(\frac{\partial p_1}{\partial B_0}\right) + \mathrm{Ad}\left[S_5(t)\right] \quad (95)$$

$$\frac{\partial B(t)}{\partial C_1} = \left(\frac{\partial S_5(t)}{\partial p_1}\right) \cdot \left(\frac{\partial p_1}{\partial B_1}\right) \quad (96)$$

$$\frac{\partial B(t)}{\partial d_0} = \frac{\partial S_5(t)}{\partial p_2} \quad (97)$$

$$\frac{\partial B(t)}{\partial d_1} = \frac{\partial S_5(t)}{\partial p_3} \quad (98)$$

$$\frac{\partial B(t)}{\partial h_0} = \frac{\partial S_5(t)}{\partial p_4} \quad (99)$$

$$\frac{\partial B(t)}{\partial h_1} = \frac{\partial S_5(t)}{\partial p_5} \quad (100)$$