Process Noise for Gaussian Random Walks

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1 Introduction: Discrete Constant Velocity Process

When performing recursive state estimation with a Kalman filter (or similar machinery), it is common to model the dynamics of the state as a Markov random walk in the highest estimated time derivatives, with zero-mean normally-distributed increments.

For instance, a constant velocity model for position $x(t)$ and velocity $v(t)$ over discrete time step $\triangle t$ is often written:

$$\begin{aligned} x(t+\triangle t) &= x(t) + \triangle t \cdot v(t) && (1)\\ v(t+\triangle t) &= v(t) + \triangle t \cdot \eta && (2)\\ \eta &\in \mathcal{N}\left(0, \triangle t \cdot \sigma_v^2\right) && (3) \end{aligned}$$

The velocity $v(t)$ wanders away from $v(0)$ with variance proportional to $t\sigma_v^2$. Figure 1 shows a few time series sampled from such a process. The velocity clearly isn't constant over time – the model is so named because the expected velocity displacement is zero from one step to the next.

Propagating the moments of a joint normal distribution of x and v through one step of the process and applying linearity of expectation yields the familiar mean and covariance updates of the Kalman filter prediction step:

$$\begin{aligned} \begin{pmatrix} x \\ v \end{pmatrix} &\in \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) && (4)\\ A(\triangle t) &\equiv \begin{pmatrix} 1 & \triangle t \\ 0 & 1 \end{pmatrix} && (5)\\ Q(\triangle t) &\equiv \begin{pmatrix} 0 & 0 \\ 0 & \triangle t \cdot \sigma_v^2 \end{pmatrix} && (6)\\ \boldsymbol{\eta} &\in \mathcal{N}(\mathbf{0}, Q) && (7)\\ \boldsymbol{\mu} &\to \mathrm{E}\left[A\boldsymbol{\mu} + \boldsymbol{\eta}\right] && (8)\\ &= A\boldsymbol{\mu} && (9) \end{aligned}$$

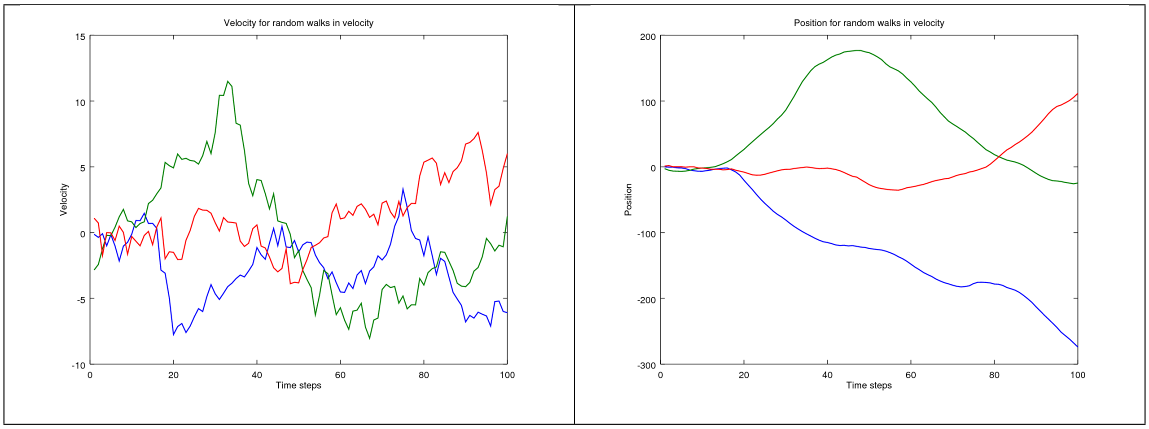

Figure 1: Sampled random walks in velocity with $\sigma_v^2 = 1$ and $\triangle t = 1$, and the corresponding time series of position.

$$\begin{aligned}
\boldsymbol{\Sigma} &\rightarrow \mathrm{E}\left[(A\boldsymbol{\mu}+\boldsymbol{\eta})(A\boldsymbol{\mu}+\boldsymbol{\eta})^T\right] && (10)\\
&= \mathrm{E}\left[(A\boldsymbol{\mu})(A\boldsymbol{\mu})^T+\boldsymbol{\eta}(A\boldsymbol{\mu})^T+(A\boldsymbol{\mu})\boldsymbol{\eta}^T+\boldsymbol{\eta}\boldsymbol{\eta}^T\right] && (11)\\
&= \mathrm{E}\left[A\boldsymbol{\mu}\boldsymbol{\mu}^T A^T\right]+\mathrm{E}\left[\boldsymbol{\eta}\boldsymbol{\mu}^T A^T\right]+\mathrm{E}\left[A\boldsymbol{\mu}\boldsymbol{\eta}^T\right]+\mathrm{E}\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\right] && (12)\\
&= A\cdot\mathrm{E}\left[\boldsymbol{\mu}\boldsymbol{\mu}^T\right]\cdot A^T+\mathrm{E}\left[\boldsymbol{\eta}\right]\cdot\boldsymbol{\mu}^T A^T+A\boldsymbol{\mu}\cdot\mathrm{E}\left[\boldsymbol{\eta}\right]^T+\mathrm{E}\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\right] && (13)\\
&= A\cdot\boldsymbol{\Sigma}\cdot A^T+Q && (14)
\end{aligned}$$

The covariance increment Q is called the *process noise*.

For later analysis, we expand the mean and covariance update expressions in terms of $\triangle t$:

$$\boldsymbol{\mu} \rightarrow \boldsymbol{\mu}+\triangle t\cdot\begin{pmatrix}\mu_v\\ 0\end{pmatrix} \quad (15)$$

$$\boldsymbol{\Sigma} \rightarrow \boldsymbol{\Sigma}+\triangle t\cdot\begin{pmatrix}2\boldsymbol{\Sigma}_{x,v}+\triangle t\cdot\boldsymbol{\Sigma}_{v,v} & \boldsymbol{\Sigma}_{v,v}\\ \boldsymbol{\Sigma}_{v,v} & \sigma_v^2\end{pmatrix} \quad (16)$$

2 Continuous Constant Velocity Process

Performing the discrete covariance propagation in Eq.16 once with step $\triangle t$ produces a different result than performing the propagation twice in a row, each with a half step $\frac{\triangle t}{2}$. That's unfortunate: the resulting variances and correlations will depend on the steps we take to get from t_0 to $t_0+\triangle t$. We'd prefer a discrete update procedure with the property that applying N steps of size $\frac{\triangle t}{N}$ yields the same result as applying one step of size $\triangle t$, for any $N > 0$. Ideally, we'd like to perform infinitely many propagation steps, each with infinitesimal step size. The result is a Wiener process in velocity (also known as Brownian motion).

With that end in mind, we reformulate the process in the continuous time domain as a system of ordinary differential equations. Differentiating Eqs. 15 and 16 by $\triangle t$ at $\triangle t = 0$ gives independent systems for the mean and covariance:

$$\dot{\boldsymbol{\mu}} = \begin{pmatrix} \boldsymbol{\mu}_v \\ 0 \end{pmatrix} \tag{17}$$

$$\dot{\boldsymbol{\Sigma}} = \begin{pmatrix} 2 \cdot \boldsymbol{\Sigma}_{x,v} & \boldsymbol{\Sigma}_{v,v} \\ \boldsymbol{\Sigma}_{v,v} & \sigma_v^2 \end{pmatrix} \tag{18}$$

These systems are straightforward to solve by definite integration over $0 \ldots t$ of first the higher and then the lower derivatives:

$$\boldsymbol{\mu}(t) = \begin{pmatrix} \boldsymbol{\mu}_x + t \cdot \boldsymbol{\mu}_v \\ \boldsymbol{\mu}_v \end{pmatrix} \tag{19}$$

$$\boldsymbol{\Sigma}_{v,v}(t) = \boldsymbol{\Sigma}_{v,v} + t \cdot \sigma_v^2 \tag{20}$$

$$\boldsymbol{\Sigma}_{x,v}(t) = \boldsymbol{\Sigma}_{x,v} + t \cdot \boldsymbol{\Sigma}_{v,v} + \frac{t^2}{2} \cdot \sigma_v^2 \tag{21}$$

$$\boldsymbol{\Sigma}_{x,x}(t) = \boldsymbol{\Sigma}_{x,x} + 2t \cdot \boldsymbol{\Sigma}_{x,v} + t^2 \cdot \boldsymbol{\Sigma}_{v,v} + \frac{t^3}{3} \cdot \sigma_v^2 \tag{22}$$

This solution can be expressed in the same form as the discrete update:

$$\boldsymbol{\mu} \rightarrow A(t) \cdot \boldsymbol{\mu} \tag{23}$$

$$= \boldsymbol{\mu} + \begin{pmatrix} t \cdot \boldsymbol{\mu}_v \\ 0 \end{pmatrix} \tag{24}$$

$$\boldsymbol{\Sigma} \rightarrow A(t) \cdot \boldsymbol{\Sigma} \cdot A(t)^T + Q(t) \tag{25}$$

$$= \boldsymbol{\Sigma} + \begin{pmatrix} 2t \cdot \boldsymbol{\Sigma}_{x,v} + t^2 \cdot \boldsymbol{\Sigma}_{v,v} & t \cdot \boldsymbol{\Sigma}_{v,v} \\ t \cdot \boldsymbol{\Sigma}_{v,v} & 0 \end{pmatrix} + Q(t) \tag{26}$$

$$Q(t) \equiv \begin{pmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix} \cdot \sigma_v^2 \tag{27}$$

The mean update for the constant velocity model is identical in the discrete and continuous formulations (Eqs. 15 and 24), because the dynamics are linear in $\boldsymbol{\mu}$, but the covariance update differs in the process noise term (Eqs. 6 and 27). Applying the continuous formulation of the process update over a finite time interval $\triangle t$ yields the same distribution regardless of how the interval is partioned into substeps.

3 Continuous Constant Acceleration Process

The same analysis applies to random walks in higher time derivatives. Consider a discrete constant acceleration model:

$$
\begin{aligned}
\begin{pmatrix} x \\ v \\ a \end{pmatrix} &\in \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) && (28)\\
A\left(\triangle t\right) &\equiv \begin{pmatrix} 1 & \triangle t & 0 \\ 0 & 1 & \triangle t \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}\left(\triangle t^2\right) && (29)\\
Q\left(\triangle t\right) &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \triangle t \cdot \sigma_a^2 \end{pmatrix} && (30)\\
\boldsymbol{\mu} &\rightarrow A\boldsymbol{\mu} && (31)\\
&= \boldsymbol{\mu} + \triangle t \cdot \begin{pmatrix} \boldsymbol{\mu}_v \\ \boldsymbol{\mu}_a \\ 0 \end{pmatrix} + \mathcal{O}\left(\triangle t^2\right) && (32)\\
\boldsymbol{\Sigma} &\rightarrow A \cdot \boldsymbol{\Sigma} \cdot A^T + Q && (33)\\
&= \boldsymbol{\Sigma} + \triangle t \cdot \begin{pmatrix} 2\boldsymbol{\Sigma}_{x,v} & \boldsymbol{\Sigma}_{v,v} + \boldsymbol{\Sigma}_{x,a} & \boldsymbol{\Sigma}_{v,a} \\ \boldsymbol{\Sigma}_{v,v} + \boldsymbol{\Sigma}_{x,a} & 2\boldsymbol{\Sigma}_{v,a} & \boldsymbol{\Sigma}_{a,a} \\ \boldsymbol{\Sigma}_{v,a} & \boldsymbol{\Sigma}_{a,a} & \sigma_a^2 \end{pmatrix} + \mathcal{O}\left(\triangle t^2\right) && (34)
\end{aligned}
$$

Differentiating the mean and covariance by $\triangle t$ at $\triangle t = 0$ gives

$$
\begin{aligned}
\dot{\boldsymbol{\mu}} &= \begin{pmatrix} \boldsymbol{\mu}_v \\ \boldsymbol{\mu}_a \\ 0 \end{pmatrix} && (35)\\
\dot{\boldsymbol{\Sigma}} &= \begin{pmatrix} 2 \cdot \boldsymbol{\Sigma}_{x,v} & \boldsymbol{\Sigma}_{v,v} + \boldsymbol{\Sigma}_{x,a} & \boldsymbol{\Sigma}_{v,a} \\ \boldsymbol{\Sigma} + \boldsymbol{\Sigma}_{x,a} & 2 \cdot \boldsymbol{\Sigma}_{v,a} & \boldsymbol{\Sigma}_{a,a} \\ \boldsymbol{\Sigma}_{v,a} & \boldsymbol{\Sigma}_{a,a} & \sigma_a^2 \end{pmatrix} && (36)
\end{aligned}
$$

Integrating over $0 \ldots t$ from higher to lower derivatives (i.e. from the lower right of $\dot{\boldsymbol{\Sigma}}$ in Eq. 36):

$$
\begin{aligned}
\boldsymbol{\mu}\left(t\right) &= \boldsymbol{\mu} + \begin{pmatrix} t \cdot \boldsymbol{\mu}_v + \frac{t^2}{2}\boldsymbol{\mu}_a \\ t \cdot \boldsymbol{\mu}_a \\ 0 \end{pmatrix} && (37)\\
\boldsymbol{\Sigma}_{a,a}\left(t\right) &= \boldsymbol{\Sigma}_{a,a} + t \cdot \sigma_a^2 && (38)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Sigma}_{v,a}(t) &= \mathbf{\Sigma}_{v,a} + t \cdot \mathbf{\Sigma}_{a,a} + \frac{t^2}{2} \cdot \sigma_a^2 & (39)\\
\mathbf{\Sigma}_{v,v}(t) &= \mathbf{\Sigma}_{v,v} + 2t \cdot \mathbf{\Sigma}_{v,a} + t^2 \cdot \mathbf{\Sigma}_{a,a} + \frac{t^3}{3} \cdot \sigma_a^2 & (40)\\
\mathbf{\Sigma}_{x,a}(t) &= \mathbf{\Sigma}_{x,a} + t \cdot \mathbf{\Sigma}_{v,a} + \frac{t^2}{2} \cdot \mathbf{\Sigma}_{a,a} + \frac{t^3}{6} \cdot \sigma_a^2 & (41)\\
\mathbf{\Sigma}_{x,v}(t) &= \mathbf{\Sigma}_{x,v} + t \cdot (\mathbf{\Sigma}_{v,v} + \mathbf{\Sigma}_{x,a}) + \frac{3t^2}{2} \cdot \mathbf{\Sigma}_{v,a} + \frac{t^3}{2} \cdot \mathbf{\Sigma}_{a,a} + \frac{t^4}{8} \cdot \sigma_a^2 & (42)\\
\mathbf{\Sigma}_{x,x}(t) &= \mathbf{\Sigma}_{x,x} + 2t \cdot \mathbf{\Sigma}_{x,v} + t^2 \cdot (\mathbf{\Sigma}_{v,v} + \mathbf{\Sigma}_{x,a}) + t^3 \cdot \mathbf{\Sigma}_{v,a} + \frac{t^4}{4} \cdot \mathbf{\Sigma}_{a,a} + \frac{t^5}{20} \cdot \sigma_a^2 & (43)
\end{aligned}
$$

Equivalently:

$$
\begin{aligned}
\mathbf{\Sigma} &\rightarrow A(t) \cdot \mathbf{\Sigma} \cdot A(t)^T + Q(t) & (44)\\
Q(t) &\equiv \begin{pmatrix} \frac{t^5}{20} & \frac{t^4}{8} & \frac{t^3}{6} \\ \frac{t^4}{8} & \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^3}{6} & \frac{t^2}{2} & t \end{pmatrix} \cdot \sigma_a^2 & (45)
\end{aligned}
$$

4 Continuous Constant Jerk Process

Applying similar analysis to a constant jerk (third time derivative) model yields:

$$
\begin{aligned}
\boldsymbol{\mu}(t) &= \boldsymbol{\mu} + \begin{pmatrix} t \cdot \boldsymbol{\mu}_v + \frac{t^2}{2}\boldsymbol{\mu}_a + \frac{t^3}{6}\boldsymbol{\mu}_j \\ t \cdot \boldsymbol{\mu}_a + \frac{t^2}{2}\boldsymbol{\mu}_j \\ t \cdot \boldsymbol{\mu}_j \\ 0 \end{pmatrix} & (46)\\
Q(t) &\equiv \begin{pmatrix} \frac{t^7}{252} & \frac{t^6}{72} & \frac{t^5}{30} & \frac{t^4}{24} \\ \frac{t^6}{72} & \frac{t^5}{20} & \frac{t^4}{8} & \frac{t^3}{6} \\ \frac{t^5}{30} & \frac{t^4}{8} & \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^4}{24} & \frac{t^3}{6} & \frac{t^2}{2} & t \end{pmatrix} \cdot \sigma_j^2 & (47)
\end{aligned}
$$

5 Continuous Damped Velocity Process

For a system where velocity is continuously and proportionally damped towards zero with strength α, the evolution of $\boldsymbol{\mu} = \begin{pmatrix} x \\ v \end{pmatrix}$ over time is a first-order system of linear differential equations:

$$
\begin{aligned}
\dot{x} &= v & (48)\\
\dot{v} &= -\alpha v & (49)
\end{aligned}
$$

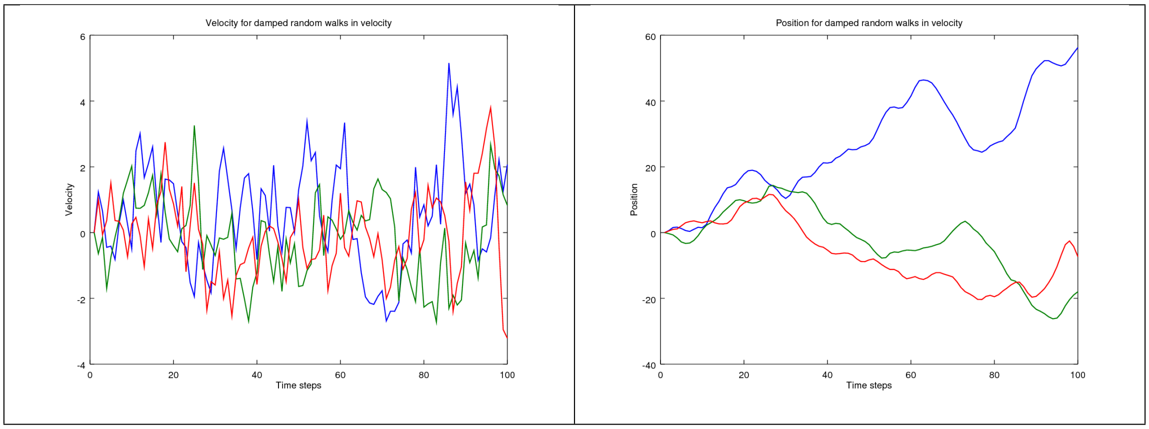

Figure 2: Sampled discrete damped velocity processes with $\alpha = 0.25$, $\sigma^2 = 1$, and $\triangle t = 1$, and the corresponding evolutions of position.

Figure 2 shows a few time series sampled from such a process.

This system can be solved by expressing the system as a matrix and exponentiating:

$$\begin{aligned}
K &\equiv \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} && (50)\\
\dot{\boldsymbol{\mu}} &= K\boldsymbol{\mu} && (51)\\
A(t) &\equiv \exp(t \cdot K) && (52)\\
&= \begin{pmatrix} 1 & \frac{1-\exp(-\alpha t)}{\alpha} \\ 0 & \exp(-\alpha t) \end{pmatrix} && (53)\\
\boldsymbol{\mu}(t) &= A(t) \cdot \boldsymbol{\mu} && (54)
\end{aligned}$$

For $\alpha \to 0$, the upper right coefficient of A should be evaluated by Taylor series expansion:

$$\frac{1-\exp(-\alpha t)}{\alpha} = t \cdot \left(1 - \frac{1}{2}\alpha t + \frac{1}{6}(\alpha t)^2 - \frac{1}{24}(\alpha t)^3 + \mathcal{O}\left(t^4\right)\right) \qquad (55)$$

The series expansion also shows that the damped velocity update is identical to the constant velocity update when $\alpha = 0$, as expected.

We can formulate the continuous covariance evolution in terms of the dynamics matrix A and the process noise Q:

$$\begin{aligned}
\dot{Q} &\equiv \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} && (56)\\
\boldsymbol{\Sigma} &\to A\boldsymbol{\Sigma}A^T + t\dot{Q} && (57)\\
&= (I + tK) \cdot \boldsymbol{\Sigma} \cdot (I + tK)^T + \mathcal{O}\left(t^2\right) + tQ && (58)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{\Sigma} + tK\mathbf{\Sigma} + t\mathbf{\Sigma}K^T + \mathcal{O}\left(t^2\right) + tQ & (59)\\
&= \mathbf{\Sigma} + t\begin{pmatrix} 2\mathbf{\Sigma}_{x,v} & \mathbf{\Sigma}_{v,v} - \alpha\mathbf{\Sigma}_{x,v} \\ \mathbf{\Sigma}_{v,v} - \alpha\mathbf{\Sigma}_{x,v} & \sigma_v^2 - 2\alpha\mathbf{\Sigma}_{v,v} \end{pmatrix} + \mathcal{O}\left(t^2\right) & (60)
\end{aligned}
$$

Differentiating by t at $t = 0$ gives:

$$
\dot{\mathbf{\Sigma}} = \begin{pmatrix} 2\mathbf{\Sigma}_{x,v} & \mathbf{\Sigma}_{v,v} - \alpha\mathbf{\Sigma}_{x,v} \\ \mathbf{\Sigma}_{v,v} - \alpha\mathbf{\Sigma}_{x,v} & \sigma_v^2 - 2\alpha\mathbf{\Sigma}_{v,v} \end{pmatrix} \tag{61}
$$

We solve the system by integrating over $0 \dots t$, starting from the lower right of Eq. 61:

$$
\begin{aligned}
\mathbf{\Sigma}(t) &= A(t) \cdot \mathbf{\Sigma} \cdot A(t)^T + Q(t) & (62)\\
Q(t) &\equiv \begin{pmatrix} \frac{4\exp(-\alpha t) - \exp(-2\alpha t) + 2\alpha t - 3}{2\alpha^3} & \frac{1}{2}\left(\frac{1-\exp(-\alpha t)}{\alpha}\right)^2 \\ \frac{1}{2}\left(\frac{1-\exp(-\alpha t)}{\alpha}\right)^2 & \frac{1-\exp(-2\alpha t)}{2\alpha} \end{pmatrix} \cdot \sigma_v^2 & (63)
\end{aligned}
$$

For $\alpha \to 0$, the coefficients of Q should be evaluated by Taylor series expansions:

$$
\begin{aligned}
\frac{1-\exp(-2\alpha t)}{2\alpha} &= t \cdot \left(1 - \alpha t + \frac{2}{3}(\alpha t)^2 - \frac{1}{3}(\alpha t)^3 + \mathcal{O}\left(t^4\right)\right) & (64)\\
\frac{1}{2}\left(\frac{1-\exp(-\alpha t)}{\alpha}\right)^2 &= \frac{t^2}{2} \cdot \left(1 - \alpha t + \frac{7}{12}(\alpha t)^2 - \frac{1}{4}(\alpha t)^3 + \mathcal{O}\left(t^4\right)\right) & (65)\\
\frac{4\exp(-\alpha t) - \exp(-2\alpha t) + 2\alpha t - 3}{2\alpha^3} &= \frac{t^3}{3} \cdot \left(1 - \frac{3}{4}\alpha t + \frac{7}{20}(\alpha t)^2 - \frac{1}{8}(\alpha t)^3 + \mathcal{O}\left(t^4\right)\right) & (66)
\end{aligned}
$$

The series expansions confirm that the damped velocity process noise is identical to the constant velocity process noise when $\alpha = 0$.